Modeling of Induction Machines with Axial Non Uniformity

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Abstract — **In this paper, a method to calculate the inductances of an induction machine with axial non uniformity is presented. The analyzed axial non-uniformities are skew and rotor eccentricities. The inductance calculation is based on a modification of the winding function theory. Theoretical fundaments of these modifications are presented. The coupled magnetic circuits approach has been used for modeling the induction machine.**

I. INTRODUCTION

The conventional d-q model of an induction machine is based on the assumption that the stator windings are sinusoidally distributed. This implies that the harmonics of the windings distribution are neglected in the analysis of the machine. A model based on the geometry and winding distribution without restrictions regarding its symmetry is more convenient for the machine analysis and simulation under asymmetric conditions.

In [1] a multiple coupled circuit machine model and a method to calculate the mutual inductances known as "*Winding Function Theory*" (WFT) are presented. By means of this model, all the harmonics of the winding distribution are taken into account, without any restrictions concerning either symmetry of stator windings or rotor bars. Hence, this model has found application in the analysis of asymmetrical and fault conditions in machines.

In [2] and [3], this method is used to analyze faults such as shorting, opening and abnormal connections of stator phase winding circuits as well as broken rotor bars and cracked rotor end-ring. The analysis of static and dynamic eccentricity effects using the cited model is presented in [4] and [5]. However, in the analysis, the equations presented in [1] that do not take into account air-gap eccentricities are used. In [6], a modification of the method for the inductance calculation considering air-gap eccentricity is proposed and applied to the analysis of dynamic eccentricity in a synchronous machine. This new method is called "*Modified Winding Function Approach*" (*MWFA*).

In the above mentioned works, the machine analysis is carried out assuming uniform eccentricity down the axial length of the motor. That is, without skew and with uniform air-gap along the rotor. In [7], the effect of skew on the inductances is analyzed using equations developed for the analysis of the machine with axial uniformity. This extension, however, does not allow the analysis of the effect produced by radial and axial air-gap non-uniformity.

In this paper, a method for the calculation of the motor windings inductances, considering radial and axial non-uniformity, is developed. Equations for inductance calculation presented in previous works can be deduced using the proposed method.

II. THE MODIFIED WINDING FUNCTION APPROACH

An elemental scheme of the machine is presented in Fig. 1 to obtain the equations that allow the induction machine inductance calculation. To make things clearer, stator windings and rotor end-rings are not shown in the scheme. There are no restrictions about the winding and rotor bar distribution and the skewing of the rotor bars for the analysis. Furthermore, restrictions over the air-gap eccentricity are not assumed. Then the machine can exhibit non-uniform either static or dynamic eccentricity down the axial length of the motor.

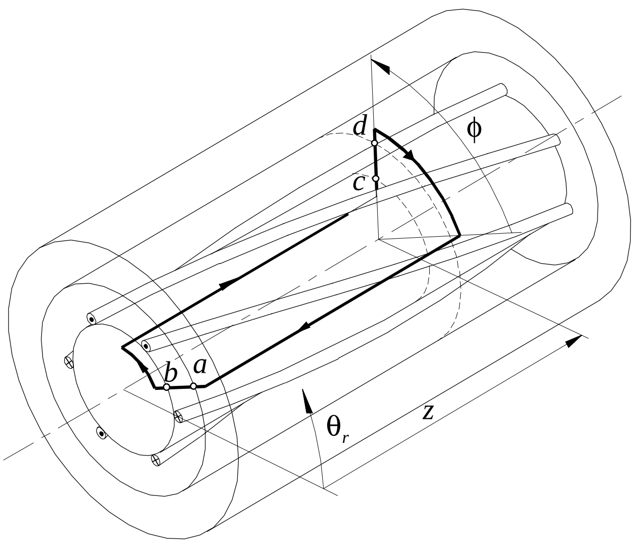

Fig. 1. Elementary Induction Machine.

The stator reference position of the closed loop *abcda,* the angle ϕ, is measured at an arbitrary point along the air-gap. The path stretches along the axial axis a length z. Points a and b are located in ϕ_0 and z_0 (both equal zero), points c and d are located in ϕ and z. On the other hand, points a and d are located on the stator internal surface whereas points b and c are located on the rotor external surface. θ_r is the rotor angle with respect to a fixed stator point.

Applying the Ampere's law over the closed path *abcda* shown in Fig. 1:

$$\oint_{abcda} H(\phi, z, \theta_r)\, dl = \int_S J\, ds, \tag{1}$$

where H is the magnetic field intensity, J is the current density and S is the surface enclosed by *abcda*. Since all the wires enclosed by the closed path carry the same current i, (1) results as follows:

$$\oint_{abcda} H(\phi, z, \theta_r)\, dl = n(\phi, z, \theta_r)\, i\ . \tag{2}$$

The function $n(\phi, z, \theta_r)$ can be called the *spatial winding distribution* and represents the number of the winding turns enclosed by the path *abcda*. This distribution, unlike previous proposes, depends on the geometry of the windings down the axial length of the motor. In terms of magnetomotive force (MMF), (2) can be written:

$$F_{ab}(0,0,\theta_r) + F_{bc} + F_{cd}(\phi, z, \theta_r) + F_{da} = n(\phi, z, \theta_r)\, i\ . \tag{3}$$

Considering infinite permeability of the stator and rotor iron cores, the MMF drops across the iron are negligible, so $F_{bc} = 0$, $F_{da} = 0$ and (3) can be re-written:

$$F_{ab}(0,0,\theta_r) + F_{cd}(\phi, z, \theta_r) = n(\phi, z, \theta_r)\, i\ . \tag{4}$$

Dividing (4) by the air-gap function $g(\phi, z, \theta_r)$, which represents the air-gap length in a given point (ϕ, z), and integrating it respect to the stator angle between 0-2π, and respect to the length of the z-axis between 0 and the rotor length (L), yields:

$$\int_0^{2\pi}\int_0^{L} \frac{F_{ab}(0,0,\theta_r)}{g(\phi, z, \theta_r)}\, dz\, d\phi + \int_0^{2\pi}\int_0^{L} \frac{F_{cd}(\phi, z, \theta_r)}{g(\phi, z, \theta_r)}\, dz\, d\phi = \int_0^{2\pi}\int_0^{L} \frac{n(\phi, z, \theta_r)\, i}{g(\phi, z, \theta_r)}\, dz\, d\phi\ . \tag{5}$$

The Gauss's Law for magnetic field is applied in order to obtain an expression for $F_{ab}(0,0,\theta_r)$,

$$\oint_S B\, ds = 0, \tag{6}$$

where B is the magnetic flux density and S is an arbitrary closed surface. Considering S as a closed cylindrical surface of radius r, placed between the stator surface and the rotor one, it results in:

$$\int_0^{2\pi}\int_0^{L} \mu_0\, r\, H(\phi, z, \theta_r)\, dz\, d\phi = 0\ . \tag{7}$$

Since,

$$H(\phi, z, \theta_r) = \frac{F(\phi, z, \theta_r)}{g(\phi, z, \theta_r)}, \tag{8}$$

then,

$$\mu_0\, r \int_0^{2\pi}\int_0^{L} \frac{F(\phi, z, \theta_r)}{g(\phi, z, \theta_r)}\, dz\, d\phi = 0\ . \tag{9}$$

Substituting (9) in (5) yields,

$$\int_0^{2\pi}\int_0^{L} \frac{F_{ab}(0,0,\theta_r)}{g(\phi, z, \theta_r)}\, dz\, d\phi = \int_0^{2\pi}\int_0^{L} \frac{n(\phi, z, \theta_r)\, i}{g(\phi, z, \theta_r)}\, dz\, d\phi\ . \tag{10}$$

Then,

$$F_{ab}(0,0,\theta_r)\, 2\pi L \left\langle g^{-1}(\phi, z, \theta_r) \right\rangle = \int_0^{2\pi}\int_0^{L} n(\phi, z, \theta_r)\, g^{-1}(\phi, z, \theta_r)\, i\, dz\, d\phi\ , \tag{11}$$

where $\left\langle g^{-1}(\phi, z, \theta_r) \right\rangle$ is the average value of the inverse air-gap function. Solving (11) for F_{ab} yields

$$F_{ab}(0,0,\theta_r) = \frac{1}{2\pi L \left\langle g^{-1}(\phi, z, \theta_r) \right\rangle} \int_0^{2\pi}\int_0^{L} n(\phi, z, \theta_r)\, g^{-1}(\phi, z, \theta_r)\, i\, dz\, d\phi. \tag{12}$$

Replacing (12) in (4) and solving for F_{cd} results in:

$$F_{cd}(\phi, z, \theta_r) = n(\phi, z, \theta_r)\, i - \frac{1}{2\pi L \left\langle g^{-1}(\phi, z, \theta_r) \right\rangle} \int_0^{2\pi}\int_0^{L} n(\phi, z, \theta_r)\, g^{-1}(\phi, z, \theta_r)\, i\, dz\, d\phi. \tag{13}$$

Dividing (13) by i, the following "Modified Winding Function" (MWF), $N(\phi, z, \theta_r)$, is obtained:

$$N(\phi, z, \theta_r) = \frac{F_{cd}(\phi, z, \theta_r)}{i}, \tag{14}$$

$$N(\phi, z, \theta_r) = n(\phi, z, \theta_r) - \frac{1}{2\pi L \left\langle g^{-1}(\phi, z, \theta_r) \right\rangle} \int_0^{2\pi}\int_0^{L} n(\phi, z, \theta_r)\, g^{-1}(\phi, z, \theta_r)\, dz\, d\phi. \tag{15}$$

The proposed MWF allows considering a no uniform geometric distribution of the windings and rotors bars down the axial length of the motor, *e.g.* skew. It is also possible to consider the rotor eccentricity effects by means of the air-gap function, $g(\phi, z, \theta_r)$, which has no restrictions about axial uniformity. This is not possible within the MWF defined in [6].

In the induction machine, the MWF can be defined for each stator winding, and each rotor loop, composed by two bars.

III. Inductance Calculations

The MMF distribution in the air-gap, produced by a current i_A flowing by any coil A is given by

$$F_A(\phi, z, \theta_r) = N_A(\phi, z, \theta_r)\, i_A \,. \tag{16}$$

So, a differential flux through a differential area $r\, dz\, d\phi$ in the air-gap will result in,

$$d\Phi = \mu_0\, F_A(\phi, z, \theta_r)\; g^{-1}(\phi, z, \theta_r)\, r\, dz\, d\phi \,, \tag{17}$$

where μ_0 is the air permeability and $g^{-1}(\phi, z, \theta_r)$ is the inverse air-gap function. Substituting (16) in (17) and integrating the differential flux in the region covered by a stator coil or rotor loop, B, yields

$$\Phi_B = \mu_0 r \int_{\phi_1}^{\phi_2} \int_{z_1(\phi)}^{z_2(\phi)} N_A(\phi, z, \theta_r)\, i_A\, g^{-1}(\phi, z, \theta_r)\; dz\, d\phi. \tag{18}$$

The total flux λ_B linking coil B is obtained from multiplying Φ_B by the turns of the coil. Since $n_B(\phi, z, \theta_r)$ is equal to the turns of the coil in the region $\{(\phi, z)/\phi_1 \le \phi \le \phi_2, z_1(\phi) \le z \le z_2(\phi)\}$, and zero otherwise, the total flux can also be obtained multiplying (17) by $n_B(\phi, z, \theta_r)$ and integrating over the whole surface,

$$\lambda_B = \mu_0 r \int_0^{2\pi} \int_0^{L} n_B(\phi, z, \theta_r)\, N_A(\phi, z, \theta_r)\, g^{-1}(\phi, z, \theta_r)\, i_A\; dz\, d\phi. \tag{19}$$

The mutual inductance L_{BA} of coil B due to the current i_A in the coil A results in,

$$L_{BA}(\theta_r) = \frac{\lambda_B}{i_A} \,, \tag{20}$$

$$L_{BA}(\theta_r) = \mu_0 r \int_0^{2\pi} \int_0^{L} n_B(\phi, z, \theta_r)\, N_A(\phi, z, \theta_r)\; g^{-1}(\phi, z, \theta_r)\, dz\, d\phi \,. \tag{21}$$

So, by means of (14) and (20) it is possible to calculate the mutual or self inductance of the stator coils or rotor loops, with or without skew. This also allows taking into account the effects of static or dynamic rotor eccentricity, without assuming uniformity down the axial length of the machine.

It is important to observe that changing the integration order and grouping terms in (20) yields,

$$L_{BA}(\theta_r) = \int_0^{L} L'_{BA}(z, \theta_r)\; dz \,, \tag{22}$$

where $L'_{BA}(z, \theta_r)$ can be defined as the mutual inductance per unit of length and is given by:

$$L'_{BA}(\theta_r) = \mu_0 r \int_0^{2\pi} n_B(\phi, z, \theta_r)\, N_A(\phi, z, \theta_r)\; g^{-1}(\phi, z, \theta_r)\, d\phi. \tag{23}$$

This result is similar to the one shown in [7]. However, the winding functions $N(\phi, z, \theta_r)$ defined in (14) are more general and allow taking into account the effects of axial air-gap non-uniformity.

IV. Calculation of Inductances with Axial Non Uniformity

A. Skew

Results of the calculation of mutual inductance between a stator phase winding and a rotor loop are shown as an example of the proposed method. Fig. 2 shows the coil distribution of phase A.

Motor parameters:
- Stator: 4 poles, 48 slots, 67 turns per coil, concentric, step 1:10:12.
- Rotor: 40 bars, skewing angle $2\pi/40$
- g = 0,45 mm

Fig. 3 shows the mutual inductance, as a function of rotor position, between the first coil of phase A, 1-12, and a rotor loop composed by two consecutive bars, R_1. In Fig. 3 a), mutual inductance without considering skew is shown, while in Fig. 3 b), a $2\pi/40$ rad. skewing is considered. From the obtained results can be observed that there is no change in the mutual inductance magnitude. However, its shape is smooth in presence of skew, mainly in the region where the inductance varies.

Mutual inductance between the phase A winding and the rotor loop is obtained by summing all the inductances between the winding coils and the rotor loop,

$$L_{AR_1}(\theta_r) = \sum_{j=1}^{8} \pm L_{A_J R_1}(\theta_r) \,. \tag{24}$$

This is the sum of the eight phase A coils.

Fig. 4 shows (thick line) the mutual inductance without considering the skew effect a), and considering it b). The derivative of the mutual inductance is also shown (thin line).

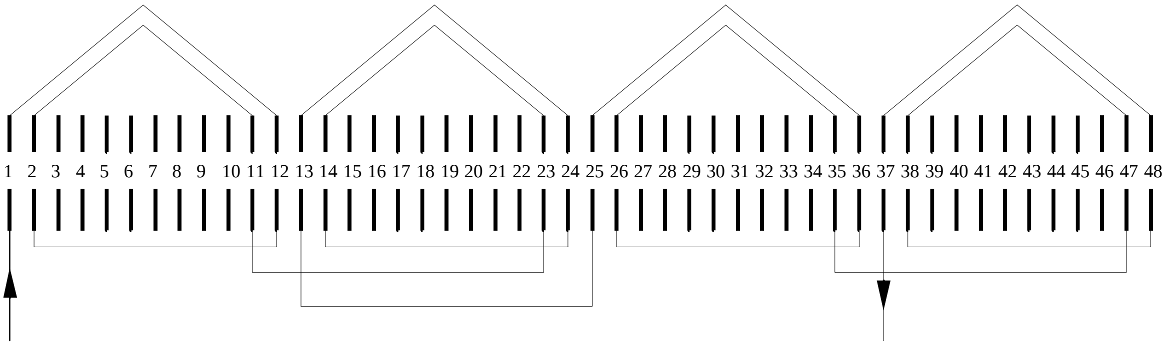

Fig. 2. Coil distribution of phase *A*.

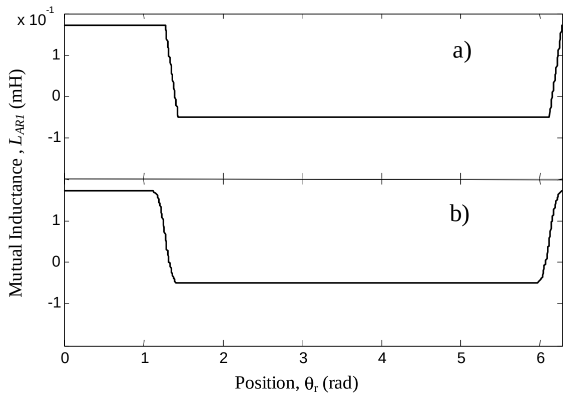

Fig. 3. Mutual Inductance of stator coil and rotor loop 1, L_{AjR1}, as a function of rotor position, θ_r. a) without skew, b) with skew.

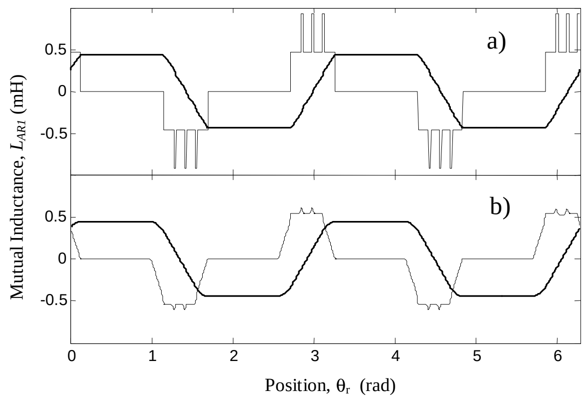

Fig. 4. Mutual Inductance of stator phase *A* and rotor loop 1, L_{AR1} as a function of rotor position, θ_r. a) without skew, b) with skew.

B. Modeling Eccentricity

As it was described in [4], there are two types of air-gap eccentricity, static and dynamic. Static eccentricity can be produced either by stator deformations or rotor axis displacement with respect to the stator axis. Due to that reason, the air-gap is non-uniform but it remains constant for any rotor position.

Dynamic eccentricity occurs when the geometric center of the rotor is not at the center of rotation, producing a periodic variation in the air-gap when the rotor turns.

In previous proposals, eccentricity effects were analyzed considering axial uniformity along the motor. The air-gap function with non-uniform static eccentricity down the axial length, g_{se}, can be represented by

$$g_{se}(\phi, z) = g_0 - \delta(z)\cos\phi \,, \tag{25}$$

where $\delta(z)$ is the air-gap variation amplitude, which can vary along the axial length of the motor. The radial air-gap length under no eccentricity condition is given by g_0.

As an example, if only one bearing is displaced respect to stator geometric axis, the air-gap variation amplitude can be approximated by:

$$\delta(z) = \delta_0 + kz \,, \tag{26}$$

where δ_0 is the air-gap variation at $z = 0$, and k is a constant.

The inverse of the air-gap function can be approximated by,

$$g_{es}^{-1}(\phi, z) = \frac{A_0(z)}{g_0} + \frac{A_1(z)}{g_0}\cos\phi \,, \tag{27}$$

where

$$A_0(z) = \frac{1}{\sqrt{1-\delta(z)}} \,, \tag{28}$$

$$A_1(z) = 2\frac{1-\sqrt{1-\delta^2(z)}}{\delta(z)\sqrt{1-\delta^2(z)}} \,. \tag{29}$$

If the eccentricity is dynamic, the air-gap function, g_{de}, can be approximated as follows,

$$g_{de}(\phi, z, \theta_r) = g_0 - \delta(z)\cos(\phi - \theta_r) \,. \tag{30}$$

These equations can be used in (14) and (20) in order to calculate both self and mutual machine inductances under eccentricity effects.

In case of static eccentricity, self and mutual inductances of stator windings are constant whereas self and mutual inductances of rotor loop change with rotor position. On the other hand, with dynamic eccentricity, self and mutual stator inductances are rotor position functions, whereas self and mutual inductances of rotor loops are not, since they do not see the air-gap change when rotor turns.

V. Modeling of Induction Machines with *m* Stator Circuits and *n* Rotor Bars.

To analyze and simulate induction machines using the inductances calculated by the proposed method, an induction machine having *m* stator circuits and *n* rotor bars is considered. The cage can be viewed as *n* identical and equally spaced rotor loops. The machine model can be obtained by neglecting saturation and eddy current losses, and supposing insulated rotor bars. This model is based on a multiple-coupled circuit approximation [1]. One might state that *m* stator circuits, and no *m* stator phases have been defined in the model. This allows to model machines with series or parallel circuits in each phase.

Voltage equations for the induction machine can be written in vector-matrix form as follows:

$$V_s = R_s\, I_s + \frac{d\lambda_s}{dt}\,, \tag{31}$$

$$V_r = R_r\, I_r + \frac{d\lambda_r}{dt}\,, \tag{32}$$

where

$$V_s = \begin{bmatrix} v_1^s & v_2^s & \ldots & v_m^s \end{bmatrix}^T, \qquad V_r = \begin{bmatrix} 0 & 0 & \ldots & 0_n \end{bmatrix}^T, \tag{33}$$

$$I_s = \begin{bmatrix} i_1^s & i_2^s & \ldots & i_m^s \end{bmatrix}^T, \qquad I_r = \begin{bmatrix} i_1^r & i_2^r & \ldots & i_n^r \end{bmatrix}^T, \tag{34}$$

and the stator and rotor flux linkages are given by

$$\lambda_s = L_{ss}\, I_s + L_{sr}\, I_r\,, \tag{35}$$

$$\lambda_r = L_{rs}\, I_s + L_{rr}\, I_r\,. \tag{36}$$

L_{ss} is an *m*m* matrix with the stator self and mutual inductances, L_{rr} is an *n*n* matrix with the rotor self and mutual inductances, L_{sr} is an *m*n* matrix composed by the mutual inductances between the stator phases and the rotor loops, L_{rs} is an *n*m* matrix composed by the mutual inductances between the rotor loops and the stator phases. All the above-mentioned inductances can be calculated using the proposed method through equations (14) and (20).

The mechanical equations for the machines are:

$$\frac{d\omega}{dt} = \frac{1}{J_{rl}}\left(T_e - T_l\right), \tag{37}$$

$$\frac{d\theta_r}{dt} = \omega\,, \tag{38}$$

where θ_r is the rotor position, ω is the angular speed and J_{rl} is the combined rotor-load inertia. T_l is the load torque and T_e is the machine electromagnetic torque which can be obtained form the magnetic co-energy W_{co},

$$T_e = \left[\frac{\partial W_{co}}{\partial \theta_r}\right]_{(I_s, I_r \text{ constant})}. \tag{39}$$

The magnetic co-energy is the energy stored in the magnetic circuits and can be written as

$$W_{co} = \frac{1}{2} I_s^T L_{ss} I_s + \frac{1}{2} I_s^T L_{sr} I_r + \frac{1}{2} I_r^T L_{rs} I_s + \frac{1}{2} I_r^T L_{rr} I_r\,. \tag{40}$$

VI. Conclusions

In this paper, a method based on a modification of the Theory of Winding Functions was developed to calculate the inductances of an induction machine. The proposed method allows taking into account radial and axial asymmetries such as skew and air-gap non-uniform eccentricity down the axial length of the motor. As an example, the skew effects on the mutual inductances between stator phases and rotor loops were shown. Equations to model static and dynamic non-uniform eccentricities were presented. In order to analyze the machine, a model considering multiple-coupled circuits with *m* stator circuits and *n* rotor bars was also presented.

VII. Acknowledgment

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