LYAPUNOV-BASED DESIGN OF ROBUST OUTPUT FEEDBACK CONTROLLERS FOR UNCERTAIN ELECTRICAL CIRCUITS

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***Abstract*— Two digital output feedback control laws are investigated in this paper aiming on the robust pole location for an electrical circuit commonly used in power electronics and supposed here as affected by uncertain parameters. One of the control laws has an unitary delay and the other is delay free. For each control law, a convex optimization problem based on a quadratic Lyapunov function is provided, allowing to find the control gains which minimize the radius of a circle, placed inside the unit circle, where the poles of the uncertain closed-loop system are located. This ensures to the closed-loop system a transient response that is the best approximation of a deadbeat response guaranteed by a quadratic Lyapunov function for the set of uncertain parameters under consideration. Numerical results illustrate the efficiency of the proposed conditions.**

***Keywords* – Convex optimization; Delayed systems; Lyapunov functions; Output feedback control; Polytopic uncertainty; Robust pole location.**

I. INTRODUCTION

The approach based on Lyapunov functions has a fundamental importance in system analysis and control. From this approach, one has that the existence of a positive function of the system state variables whose time-derivative is negative (i.e. a Lyapunov function) ensures the system (open-loop or closed-loop) stability [8]. The problem of searching a Lyapunov function, which is the main difficulty with this methodology, can be regarded for many systems as a problem of solving linear matrix inequalities [4], for which available interior point based algorithms, as for instance [6], provide a solution in polynomial time, whenever a solution exists. Since the linear matrix inequalities used to search a Lyapunov function must take into account the system dynamics, the solution for the problem becomes more involved when the system model incorporates, for instance, uncertain parameters and nonlinearities [1, 3, 5, 8, 10–16].

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This paper aims on the use of quadratic Lyapunov functions to obtain the gains of two digital output feedback control laws applied to ensure robust pole location for a second order electrical circuit commonly used in power electronics applications. Differently from usual approaches, the circuit parameters are supposed here as uncertain parameters. The first control law is not affected by delay and the second control law is affected by an unitary delay on feedback. For each one of the control laws, the design problem is to find the control gains which minimize the radius of a circle centered at the origin of the complex plane and included in the unit circle which contains all the poles of the closed-loop uncertain system. This problem can be recast as a convex optimization problem for which the proposed conditions allow to get the control gains which provide the global minimum value of the radius of the circle for pole location under quadratic stability. Numerical results illustrate the efficiency of the conditions given in the paper.

II. SYSTEM MODELING

Consider the electrical circuit in Figure 1, which is used as a stage in several power electronics applications, as for instance in uninterruptible power supplies, in AC power sources and in DC-DC converters [7].

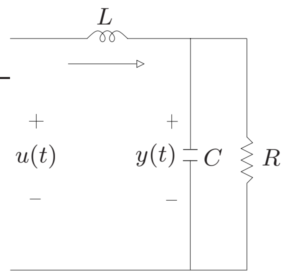

Fig. 1. Second order electrical circuit.

This circuit is the plant to be controlled here by means of digital control techniques, where $u(t)$ is the control input, driven by a switching power electric circuit, considered as the actuator of the system, and $y(t)$ is the controlled output. Differently from usual approaches, in this paper the parameters L (inductance), C (capacitance) and R (resistance) are assumed as uncertain parameters belonging to real compact intervals given by

$$\begin{array}{c} L \in \Omega_L = \{L \in \mathbb{R}_+^\star : \underline{L} \le L \le \overline{L}\} \\ C \in \Omega_C = \{C \in \mathbb{R}_+^\star : \underline{C} \le C \le \overline{C}\} \\ R \in \Omega_R = \{R \in \mathbb{R}_+^\star : \underline{R} \le R \le \overline{R}\} \end{array} \quad (1)$$

for which the lower and upper bounds are known.

The transfer function of the system in Figure 1 is given by

$$\frac{Y(s)}{U(s)} = \frac{w_n^2}{s^2 + 2\xi w_n s + w_n^2} \ ,$$
$$w_n = \frac{1}{\sqrt{LC}} \ , \ \xi = \frac{1}{2R}\sqrt{\frac{L}{C}} \quad (2)$$

where w_n and ξ also belong to real compact intervals.

Assuming that $u(t)$ is the average value of the voltage pulse produced by the actuator during the sampling period T, one obtains, by means of the zero-order hold method [2], the following discrete-time model of the plant

$$\frac{Y(z)}{U(z)} = \frac{b_1 z + b_0}{z^2 + a_1 z + a_0} \quad (3)$$

where

$$a_1 = -2\alpha\beta \ , \ a_0 = \alpha^2 \ , \ b_1 = 1 - \alpha(\beta + \xi\frac{w_n}{w_1}\gamma) \ ,$$
$$b_0 = \alpha^2 + \alpha(\xi\frac{w_n}{w_1}\gamma - \beta) \ , \ w_1 = w_n\sqrt{1-\xi^2} \ ,$$
$$\alpha = \exp(-\xi w_n T) \ , \ \beta = \cos(w_1 T) \ , \ \gamma = \sin(w_1 T) \quad (4)$$

Notice that[1] $\forall(L, C, R) \in \{\Omega_L \times \Omega_C \times \Omega_R\} \triangleq \mathcal{H}_c$, one has that $(a_0, a_1, b_0, b_1) \in \mathcal{H}_d$, where $\mathcal{H}_d$ is the subset of $\mathbb{R}^4$ obtained from the nonlinear functions (4) applied to every $(L, C, R) \in \mathcal{H}_c$.

The transfer function (3) admits the space state representation

$$x(k+1) = A_d(a_0, a_1)x(k) + Bu(k) \quad (5)$$

$$y(k) = C_d(b_0, b_1)x(k) \quad (6)$$

where $x(k) = [x_1(k) \quad x_2(k)]'$,

$$A_d(a_0, a_1) = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \ ,$$
$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ , \ C_d(b_0, b_1) = \begin{bmatrix} b_0 & b_1 \end{bmatrix}$$

[1]The symbol $\times$ between intervals represents Cartesian product and $\triangleq$ means equal by definition.

and $(a_0, a_1, b_0, b_1) \in \mathcal{H}_d$.

Observe also that $\forall(L, C, R) \in \mathcal{H}_c$, it follows that

$$\begin{array}{l} a_1 \in \Omega_{a_1} = \{a_1 \in \mathbb{R} : \underline{a_1} \le a_1 \le \overline{a_1}\} \\ a_0 \in \Omega_{a_0} = \{a_0 \in \mathbb{R} : \underline{a_0} \le a_0 \le \overline{a_0}\} \\ b_1 \in \Omega_{b_1} = \{b_1 \in \mathbb{R} : \underline{b_1} \le b_1 \le \overline{b_1}\} \\ b_0 \in \Omega_{b_0} = \{b_0 \in \mathbb{R} : \underline{b_0} \le b_0 \le \overline{b_0}\} \end{array}$$

which allows to include system (5)-(6) in the polytopic system [4]

$$x(k+1) = A(\alpha)x(k) + Bu(k) \quad (7)$$

$$y(k) = C(\alpha)x(k) \quad (8)$$

where

$$A(\alpha) = \sum_{j=1}^{16} \alpha_j A_j \ , \ C(\alpha) = \sum_{j=1}^{16} \alpha_j C_j$$

and A_j and C_j, $j = 1, \ldots, 16$ are the vertices of the polytopes of matrices $A(\alpha)$ and $C(\alpha)$. The vector of uncertain parameters, α, belongs to the unit simplex

$$\mathcal{U} = \{\alpha \in \mathbb{R}^{16} \ : \ \sum_{j=1}^{16} \alpha_j = 1 \ , \ \alpha_j \ge 0 \ , \ j = 1, \ldots, 16\} \quad (9)$$

Remark 1: Any property valid for the polytopic system (7)-(8) is also valid for system (5)-(6), since (5)-(6) is included in (7)-(8).

III. PROBLEM FORMULATION

The main objective of this paper is to solve problems 1 and 2, described in the sequel.

Problem 1: Consider the output feedback control law

$$u(k) = k_0 y(k) \quad (10)$$

which allows to write the control system (7)-(8) as

$$x(k+1) = A_{cl}(\alpha, k_0)x(k) \quad (11)$$

where

$$A_{cl}(\alpha, k_0) = \sum_{j=1}^{16} \alpha_j A_{clj}(k_0) \ , \ \alpha \in \mathcal{U} \ ,$$
$$A_{clj}(k_0) = A_j + k_0 B C_j \ , \ j = 1, \ldots, 16$$

Find $k_0 \in \mathcal{K}_0$, where

$$\mathcal{K}_0 \triangleq \{\underline{k_0}, \underline{k_0} + \delta, \ldots, \overline{k_0} - \delta, \overline{k_0}\} \quad (12)$$

for which there exists a quadratic Lyapunov function

$$v(x(k)) = x(k)' P x(k) \quad (13)$$

ensuring the solution for the optimization problem[2]

$$\begin{array}{c} r_{P1}^* \triangleq \min_{k_0 \in \mathcal{K}_0} r \text{ s.t.} \\ 0 < r \le 1 \\ \max|\lambda(A_{cl}(\alpha, k_0))| < r \ , \ \forall \alpha \in \mathcal{U} \end{array} \quad (14)$$

[2]$\lambda(.)$ represents the operator which extracts the eigenvalues of a square matrix.

Remark 2: Notice from Problem 1 that one must find k_0 in a finite set, defined by $\underline{k_0}$, $\overline{k_0}$ and δ. These values can be based, for instance, on the limits of precision of the digital platform used to implement the controller.

Remark 3: The optimization problem described in Problem 1 can be read as find $k_0 \in \mathcal{K}_0$ which minimizes the upper bound for the maximum absolute value of the closed-loop poles, based on quadratic stability. The solution for this problem provides a controller which ensures the minimum upper bound for the settling time of the state trajectories in the transient response of the closed-loop system based on the quadratic Lyapunov function (13). Moreover, it is clear that the solution of (14) ensures that $\max|\lambda(A_{cl}(\alpha,k_0))| < 1$, $\forall\alpha \in \mathcal{U}$, thus guaranteeing the global asymptotic stability of the closed-loop system, that is, for any initial condition $x(0)$, $x(\infty) = \mathbf{0}$.

Remark 4: From Remark 3, one has that the solution for (14) guarantees the location of the poles of the closed-loop system inside a circle included in the unit circle for the entire set of uncertain parameters, that is, the solution for (14) provides robust pole location with respect to the uncertain parameters $\alpha \in \mathcal{U}$.

Problem 2: Consider the output feedback control law with unitary delay

$$u(k) = k_1 y(k-1) \tag{15}$$

which allows to write the control system (7)-(8) as

$$\tilde{x}(k+1) = \tilde{A}_{cl}(\alpha,k_1)\tilde{x}(k) \tag{16}$$

where $\tilde{x}(k) = [x_0(k)\ x_1(k)\ x_2(k)]'$ and

$$\tilde{A}_{cl}(\alpha,k_1) = \sum_{j=1}^{16} \alpha_j \tilde{A}_{clj}(k_1)\ ,\ \alpha \in \mathcal{U}\ ,$$
$$\tilde{A}_{clj}(k_1) = \tilde{A}_j + k_1 \tilde{B}\tilde{C}_j\ ,\ j = 1,\ldots,16$$

with

$$\tilde{A}_j = \begin{bmatrix} 0 & L \\ M & A_j \end{bmatrix}\ ,\ L = \begin{bmatrix} 1 & 0 \end{bmatrix}\ ,$$
$$M = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\ ,\ \tilde{B} = \begin{bmatrix} 0 \\ B \end{bmatrix}\ ,\ \tilde{C}_j = \begin{bmatrix} 0 & C_j \end{bmatrix}$$

Find $k_1 \in \mathcal{K}_1$, where

$$\mathcal{K}_1 \triangleq \{\underline{k_1}, \underline{k_1} + \delta, \ldots, \overline{k_1} - \delta, \overline{k_1}\} \tag{17}$$

for which there exists a quadratic Lyapunov function

$$v(\tilde{x}(k)) = \tilde{x}(k)'\tilde{P}\tilde{x}(k) \tag{18}$$

ensuring the solution for the optimization problem:

$$\begin{gathered} r^*_{P2} \triangleq \min_{k_1 \in \mathcal{K}_1} r \text{ s.t.} \\ 0 < r \leq 1 \\ \max|\lambda(\tilde{A}_{cl}(\alpha,k_1))| < r\ ,\ \forall\alpha \in \mathcal{U} \end{gathered} \tag{19}$$

Remarks similar to remarks 2 to 4 are applicable to Problem 2.

Remark 5: Observe that due to the unitary delay on the feedback, which can arise for instance from measurement and processing delay, the closed-loop system (16) requires one more state variable to be described when compared to the closed-loop system for the system without delay (11).

IV. MAIN RESULTS

Solutions for problems 1 and 2 are given in this section, based on the solution of convex optimization problems called generalized eigenvalue problems [4]. This class of optimization problems has the great advantage of being solvable by globally convergent algorithms, as for instance [6], which provide the global optimal solution in polynomial time.

Theorem 1: Given $k_0 \in \mathcal{K}_0$. If, and only if, there exists a solution for the following convex optimization problem:

$$\begin{gathered} r^*(k_0) \triangleq \min_{P=P' \in \mathbb{R}^{2\times 2}} r \\ \text{s.t.} \\ 0 < r \leq 1 \\ \begin{bmatrix} rP & A_{clj}(k_0)'P \\ PA_{clj}(k_0) & rP \end{bmatrix} > 0\ ,\ j = 1,\ldots,16 \end{gathered} \tag{20}$$

then (13) is a quadratic Lyapunov function ensuring that $\max|\lambda(A_{cl}(\alpha,k_0))| < r^*(k_0)$, for all $\alpha \in \mathcal{U}$.

Proof: For a given $k_0 \in \mathcal{K}_0$ and for a given $0 < r \leq 1$, one has that the existence of $P = P' > 0$ solving the linear matrix inequality

$$\frac{A_{cl}(\alpha,k_0)'}{r} P \frac{A_{cl}(\alpha,k_0)}{r} - P < 0\ ,\ \forall\alpha \in \mathcal{U} \tag{21}$$

is necessary and sufficient for $\max|\lambda(A_{cl}(\alpha,k_0))| < r$, for all $\alpha \in \mathcal{U}$, under quadratic stability [9]. By applying Schur complement and by taking into account the convexity of $\mathcal{U}$, one has that the matrix inequality in (20) is necessary and sufficient to ensure (21). The global minimum value of r solving the matrix inequality in (20), defined as $r^*(k_0)$, can be obtained by means of the solution of the generalized eigenvalue problem (20) [4]. ■

Corollary 1: From all $k_0 \in \mathcal{K}_0$ which fulfill (20), the solution of Problem 1 is given by k_0 which provide the minimum value of $r^*(k_0)$, called r^*_{P1}.

Theorem 2: Given $k_1 \in \mathcal{K}_1$. If, and only if, there exists a solution for the following convex optimization problem:

$$\begin{gathered} r^*(k_1) \triangleq \min_{\tilde{P}=\tilde{P}' \in \mathbb{R}^{3\times 3}} r \\ \text{s.t.} \\ 0 < r \leq 1 \\ \begin{bmatrix} r\tilde{P} & \tilde{A}_{clj}(k_1)'\tilde{P} \\ \tilde{P}\tilde{A}_{clj}(k_1) & r\tilde{P} \end{bmatrix} > 0\ ,\ j = 1,\ldots,16 \end{gathered} \tag{22}$$

then (18) is a quadratic Lyapunov function ensuring that $\max|\lambda(\tilde{A}_{cl}(\alpha, k_1))| < r^*(k_1)$, for all $\alpha \in \mathcal{U}$.

Proof: Follows the ideas in the proof of Theorem 1. ■

Corollary 2: From all $k_1 \in \mathcal{K}_1$ which fulfill (22), the solution of Problem 2 is given by k_1 which provide the minimum value of $r^*(k_1)$, called r^*_{P2}.

V. NUMERICAL RESULTS

The next examples present a comparison between both control laws in the solution of problems 1 and 2. In the first example, only R is considered as an uncertain parameter in the plant, lying on a wide interval. In the second example, L, C and R are supposed as uncertain parameters with 10% of perturbation around their nominal values. For both examples, the sampling period is $T = 1/1800\ s$ and the sets $\mathcal{K}_0$ and $\mathcal{K}_1$ are given by $\{-1, -1+0.01, \ldots, 1\}$.

Example 1: Consider that the parameters of the plant are given by: $24\Omega \leq R \leq 1M\Omega$, $L = 3\ mH$ and $C = 120\mu F$. The solution for Problem 1, provided by Corollary 1, is $r^*_{P1} = 0.99$, being this optimal value guaranteed by any of the values of k_0 in the set $\{0.4, 0.4+0.01, \ldots, 0.67\}$. The solution for Problem 2, provided by Corollary 2, is $r^*_{P2} = 0.77$, for the values of k_1 in the set $\{0.52, 0.53\}$. Notice the considerable improvement on the minimization of the upper bound of the maximum absolute value of the closed-loop poles based on quadratic stability provided by the control law (15) (with delay) when compared to the control law (10) (without delay) for this example.

Example 2: Define $R_n = 24\Omega$, $L_n = 3\ mH$ and $C_n = 120\mu F$ and consider $0.9R_n \leq R \leq 1.1R_n$, $0.9L_n \leq L \leq 1.1L_n$ and $0.9C_n \leq C \leq 1.1C_n$. Corollary 1 does not provide any solution for Problem 1 in this case. On the other hand, Corollary 2 yields $r^*_{P2} = 0.83$ as a solution for Problem 2 for any value of k_1 belonging to the set $\{0.44, 0.45, 0.46\}$, thus illustrating again the superiority of control law (15) (with delay) to solve the problem of minimization of the upper bound of the maximum absolute value of the closed-loop poles under quadratic stability.

VI. CONCLUSION

This paper investigates the stabilization of a plant given by a second order electrical circuit with uncertain parameters by means of two digital output feedback control laws: $u(k) = k_0 y(k)$ (without delay) and $u(k) = k_1 y(k-1)$ (with delay). For each control law, a convex optimization problem is proposed, allowing to find the value of the control gain inside previously given finite sets which provides the global minimum upper bound for the maximum absolute value of the closed-loop poles under quadratic stability for the uncertain system (robust pole location). Numerical comparisons illustrate that the control law subject to delay provides better robust pole location under quadratic stability. The proposed conditions can also be used to solve the output feedback design problems given here when the system is not affected by uncertainties, by simply employing fixed matrices $A_{cl}(k_0)$ and $\tilde{A}_{cl}(k_1)$ in the related theorems, instead of sets of matrices. The proposed synthesis conditions can also be applied as a robust stabilization stage in problems of regulation and tracking. Finally, although the results have been derived for a specific electrical circuit and for unitary delay, the methodology can be extended to other circuits with uncertain parameters as well as to cope with multiple delays affecting the output feedback.

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